Local uniqueness for the Dirichlet-to-Neumann map via the two-plane transform

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Abstract

We consider the Cauchy data associated to the Schrödinger equation with a potential on a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$. We show that the integral of the potential over a two-plane Π is determined by the Cauchy data of certain exponentially growing solutions on any open subset $\mathcal{U} \subset \partial\Omega$ which contains $\Pi \cap \partial\Omega$.

0 Introduction

For Ω a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $\partial\Omega$, and real-valued $q(x) \in L^\infty(\Omega)$, let

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \tag{0.1}$$

be the Dirichlet-to-Neumann map associated with the operator $\Delta + q$ on Ω, which is defined if $\lambda = 0$ is not a Dirichlet eigenvalue for $\Delta + q$ on Ω. More generally, one may consider the set of Cauchy data of solutions of

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$(\Delta + q(x))v = 0$, which is defined even if $\lambda = 0$ is a Dirichlet eigenvalue. Set

(0.2)
$$\mathcal{CD}_q = \Big\{(v|_{\partial\Omega}, \frac{\partial v}{\partial n}|_{\partial\Omega}) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) : v \in H^1(\Omega), (\Delta + q)v = 0\Big\},$$

which is a subspace of $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$; if Λ_q is defined, then $\mathcal{CD}_q$ is simply the graph of Λ_q.

This paper is concerned with the problem of obtaining partial knowledge of $q(x)$ from partial knowledge of $\mathcal{CD}_q$, namely its restriction to certain "small" open subsets of the boundary. The approach taken here is to use concentrated, exponentially growing, approximate solutions to relate $\mathcal{CD}_q$ on an open set $\mathcal{U} \subset \partial\Omega$ to the two-plane transform of the potential $q(x)$ on two-planes whose intersections with $\partial\Omega$ are contained in $\mathcal{U}$.

Let $M_{2,n}$ denote the $(3n-6)$-dimensional Grassmannian of all affine two-planes $\Pi \subset \mathbb{R}^n$, and

$$R_{2,n}f(\Pi) = \int_\Pi f(y)d\lambda_\Pi(y), f \in L^2(\mathbb{R}^n), \tag{0.3}$$

denote the two-plane transform on $\mathbb{R}^n$ [H65, H80]. Here, $d\lambda_\Pi$ is two-dimensional Lebesgue measure on $\Pi \in M_{2,n}$, which can be defined by

$$< f, d\lambda_\Pi >= \lim_{\epsilon \to 0} \frac{1}{|B^{n-2}(0;\epsilon)|} \int_{\{dist(x,\Pi)<\epsilon\}} f(x)dx. \tag{0.4}$$

(Note that for $n = 3$, $R_{2,3}$ is just the usual Radon transform on $\mathbb{R}^3$.) We will also need the variant of $d\lambda_\Pi$ defined relative to Ω:

$$< f, d\lambda_\Pi^\Omega >= \lim_{\epsilon \to 0} \frac{1}{|B^{n-2}(0;\epsilon)|} \int_{\Omega \cap \{dist(x,\Pi)<\epsilon\}} f(x)dx, \tag{0.5}$$

which gives rise to a two-plane transform relative to Ω,

$$R_{2,n}^\Omega f(\Pi) = \int_\Pi f(x)d\lambda_\Pi^\Omega(x). \tag{0.6}$$

Note that if $\partial\Omega$ is C^1 and $\Pi \cap \partial\Omega$ transversally, then $< d\lambda_\Pi^\Omega, f >= < d\lambda_\Pi, f \cdot \chi_\Omega >$ and $R_{2,n}^\Omega f(\Pi) = R_{2,n}(f \cdot \chi_\Omega)(\Pi)$.

For each choice of an orthonormal basis for Π_0, the translate of Π passing through the origin, as well as other arbitrary choices made below, we will construct a family, $\mathcal{F}_q = \{v_z(x) : z \in \mathbb{C}, |z| \geq C\}$, of exponentially growing

solutions of $(\Delta + q(x))v = 0$, concentrated near Π. Using these families, we formulate

Definition (i) If $\mathcal{U} \subset \partial\Omega$ is open, $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are *equal on* $\mathcal{U}$ *relative to* $\mathcal{F}$ *at* $z \in \mathbb{C}$ if the solutions in $\mathcal{F}_{q_1}$ and $\mathcal{F}_{q_2}$ corresponding to opposite exponential growths, $v_z^{(1)}$ and $v_{-z}^{(2)}$, have the same Cauchy data on $\mathcal{U}$:

$$(v_z^{(1)}|_{\mathcal{U}}, \frac{\partial v_z^{(1)}}{\partial n}|_{\mathcal{U}}) = (v_{-z}^{(2)}|_{\mathcal{U}}, \frac{\partial v_{-z}^{(2)}}{\partial n}|_{\mathcal{U}}).$$

(ii) $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are *equal on* $\mathcal{U}$ *for a sequence of exponentially growing solutions* if $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are equal on $\mathcal{U}$ relative to $\mathcal{F}$ at $z = z_j$ for some sequence $\{z_j\}_1^\infty \subset \mathbb{C}$ with $|z_j| \to \infty$.

We may now state the main result proved here. For each $\Pi \in M_{2,n}$, let $\gamma_\Pi = \Pi \cap \partial\Omega \subset \partial\Omega$, and let $H^s(\Omega)$ denote the standard Sobolev space of distributions with s derivatives in $L^2(\Omega)$.

Theorem 1 *Let $n \geq 3$. Assume $\partial\Omega$ is Lipschitz and potentials $q_1(x)$ and $q_2(x)$ are in $H^s(\Omega)$, for some $s > \frac{n}{2}$. Let $\Pi \in M_{2,n}$ and $\mathcal{F}_{q_1}$ and $\mathcal{F}_{q_2}$ be families of exponentially growing solutions associated to q_1 and q_2. If, for some fixed neighborhood $\mathcal{U}_\Pi$ of γ_Π in $\partial\Omega$, $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are equal on $\mathcal{U}_\Pi$ for a sequence of exponentially growing solutions, then*

$$R^\Omega_{2,n}(q_1 - q_2)(\Pi) = 0, \tag{0.7}$$

i.e., $\int q_1(y) d\lambda^\Omega_\Pi(y) = \int q_2(y) d\lambda^\Omega_\Pi(y)$.

If $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ equal on all of $\partial\Omega$ relative to $\mathcal{F}$, then this implies that $R_{2,n}((q_1 - q_2)\chi_\Omega)(\Pi) = 0$, $\forall\ \Pi \in M_{2,n}$, which by the uniqueness theorem for $R_{2,n}$ yields that $q_1 - q_2 \equiv 0$ on Ω, providing a variant of the global uniqueness theorem for the Dirichlet–to–Neumann map [SU87a]. (We note that our technique is limited to three or more dimensions and says nothing in the case $n = 2$ [N96].) However, one is also able to obtain *local* uniqueness results by replacing the uniqueness theorem for the two-plane transform with Helgason's support theorem [H80, Cor. 2.8]: if $C \subset \mathbb{R}^n$ is a closed, convex set and $f(x)$ a function[1] such that $R_{2,n}f(\Pi) = 0$ for all Π disjoint from C, then $\mathrm{supp}(f) \subset C$. We then immediately obtain the following two results.

[1] The support and uniqueness theorems are usually stated under the assumption that $f(x)$ is continuous, of rapid decay in the case of the support theorem, but the proofs in [H80] are easily seen to extend to the case where $f(x) = q(x)\chi_\Omega(x)$ with $\Omega \subset \mathbb{R}^n$ bounded, $q \in C(\overline{\Omega})$.

Theorem 2 *Suppose $\partial\Omega$ and potentials q_1, q_2 are as in Thm. 1., and $C \subset \Omega$ is a closed, convex set. If, for all $\Pi \in M_{2,n}$ such that $\Pi \cap C = \phi$, there is some neighborhood $\mathcal{U}_\Pi$ of γ_Π on which $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are equal for some sequence of exponentially growing solutions, then* $\mathrm{supp}(q_1 - q_2) \subseteq C$, *i.e., $q_1 = q_2$ on $\Omega \backslash C$.*

Theorem 3 *Suppose $\partial\Omega$ is C^2 and strictly convex, and potentials q_1, q_2 are as in Thm. 1. If, for some $r > 0$, $\mathcal{CD}_{q_1}$ and $\mathcal{CD}_{q_2}$ are equal on B for some sequence of exponentially growing solutions for all surface balls $B = B^n(x_0; r) \cap \partial\Omega \subset \partial\Omega$, then*

$$\mathrm{dist}(\mathrm{supp}(q_1 - q_2), \partial\Omega) \geq Cr^2,$$

i.e., $q_1 = q_2$ on the tubular neighborhood $\{x \in \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) \leq Cr^2\}$ of $\partial\Omega$ in $\overline{\Omega}$.

Remark

The conclusions of Thms. 2 and 3 can be strengthened by combining them with a result in Isakov [Is]. Namely, if either $C \subset\subset \Omega$ in Thm. 2, or the assumption of Thm. 3 holds for some $r > 0$, we can conclude from Thm. 2 or 3 that $supp(q_1 - q_2) \subset\subset \Omega$. By Ex. 5.7.4 in [Is], based on a technique of Kohn and Vogelius[KV85], this, together with the condition that $\Lambda_{q_1} = \Lambda_{q_2}$ on some open set $\mathcal{U} \subset \partial\Omega$, implies that $q_1 \equiv q_2$ everywhere on Ω. We are indebted to Adrian Nachman for pointing this out to us.

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1 Approximate solutions

To prove Thm. 1, we first construct exponentially growing *approximate* solutions for $(\Delta + q)v = 0$. As considered in [C, SU86, SU87a], let

$$\mathcal{Q} = \{\rho \in \mathbb{C}^n : \rho \cdot \rho = 0\}$$

be the (complex) characteristic variety of Δ. Each $\rho \in \mathcal{Q}$ can be written as $\rho = |\rho|\frac{\rho}{|\rho|} = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in \mathbb{R} \cdot (S^{n-1} + iS^{n-1})$, with $\omega_R \cdot \omega_I = 0$. For $\rho \in \mathcal{Q}$, let $\Delta_\rho = \Delta + 2\rho \cdot \nabla$. Then

$$\Delta_\rho + q(x) = e^{-\rho \cdot x}(\Delta + q(x))e^{\rho \cdot x}, \tag{1.1}$$

so that, with $v(x) = e^{\rho\cdot x}u(x)$,

$$(1.2) \qquad (\Delta_\rho + q(x))u(x) = w(x) \Leftrightarrow (\Delta + q(x))v(x) = e^{\rho\cdot x}w(x)$$

and, in particular, $(\Delta_\rho + q(x))u(x) = 0 \Leftrightarrow (\Delta + q(x))v(x) = 0$.

Now, given a potential $q(x)$ and a two-plane $\Pi \in M_{2,n}$, we will construct an approximate solution u_{app} to $(\Delta_\rho + q)u = 0$, supported near Π:

Theorem 4 *Let Ω be Lipschitz and $q(x) \in H^s(\Omega)$ for some $s > \frac{n}{2}$. Then, for any $0 < \beta < \frac{1}{4}$ fixed, the following holds: $\exists\ \epsilon > 0$ such that, for any $\rho = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in \mathcal{Q}$ and any two-plane Π parallel to $\Pi_0 = \mathrm{span}\{\omega_R, \omega_I\}$, we can find an approximate solution $u_{app} = u_{app}(x, \rho, \Pi)$ to $(\Delta_\rho + q(x))u = 0$ satisfying*

$$(1.5) \qquad \|u_{app}\|_{L^2(\mathbb{R}^n)} \leq C, \quad \|u_{app}\|_{L^2(\Omega)} \simeq [\lambda_\Pi^\Omega(\Pi \cap \Omega)]^{\frac{1}{2}} \ \text{as } |\rho| \to \infty$$

$$(1.6) \qquad \mathrm{supp}(u_{app}) \subset \left\{ x \in \mathbb{R}^n : \mathrm{dist}(x, \Pi) \leq \frac{2}{|\rho|^\beta} \right\}$$

and

$$(1.7) \qquad \|(\Delta_\rho + q)u_{app}\|_{L^2(\mathbb{R}^n)} \leq \frac{C_\epsilon}{|\rho|^\epsilon}.$$

Furthermore, for any two such solutions, $u_{app}^{(1)}, u_{app}^{(2)}$, associated with possibly different potentials $q_1(x), q_2(x)$ and with $\rho_1 \in \mathcal{Q}, \rho_2 = e^{i\theta}\rho_1$ or $\rho_2 = e^{i\theta}\overline{\rho_1} \in \mathcal{Q}$,

$$(1.8) \qquad u_{app}^{(1)}(\cdot, \rho_1, \Pi)u_{app}^{(2)}(\cdot, \rho_2, \Pi) \to d\lambda_\Pi^\Omega \ \text{weakly as } |\rho_1| \to \infty.$$

In fact, as will be seen below, $u_{app} = u_0 + u_1$ with u_0 depending only on Π and $|\rho|$ and satisfying (1.5).

Now, we may apply the results of [SU86, SU87a] (see also [Ha96]) to find a solution u_2 of

$$(\Delta_\rho + q)u_2 = -(\Delta_\rho + q)u_{app} \in L^2_{comp}(\mathbb{R}^n),$$

uniformly in H^1_t and with a gain of $|\rho|^{-1}$ in L^2_t, as long as $|\rho| \geq C$ with C depending only on $\|q\|_\infty$ and $diam(\Omega)$. Here, H^s_t and L^2_t the weighted versions of these spaces, as in [SU87a], for some fixed $-1 < t < 0$. By these results and (1.7),

$$\|u_2\|_{H^1_t(\mathbb{R}^n)} \leq c\|(\Delta_\rho + q)u_{app}\|_{L^2_{t+1}(\mathbb{R}^n)} \leq c|\rho|^{-\epsilon}, \quad \|u_2\|_{L^2_t} \leq C|\rho|^{-1-\epsilon}.$$

(The statements in [SU86,SU87a] are for $q \in C^\infty$, but the proofs are easily seen to hold if $q \in H^s(\Omega)$ with $s > \frac{n}{2}$. Also, the weights will be irrelevant since we will be working on Ω.) Thus, $u = u_{app} + u_2 = u_0 + u_1 + u_2$ is an exact solution of $(\Delta_\rho + q)u = 0$ on $\mathbb{R}^n$, satisfying

$$\|u - u_0\|_{L^2} \leq c|\rho|^{-\epsilon} \text{ and } \|u_2\|_{H^s} \leq |\rho|^{s-1-\epsilon}, \forall 0 \leq s \leq 1.$$

Finally,

$$\mathcal{F}_q = \{v_z : |z| \geq C\} = \{e^{\rho\cdot x}u(x,\Pi,\rho) : \rho = Re(z)\omega_R + iIm(z)\omega_I, |z| \geq C\}$$

is the associated family of exponentially growing solutions used in the statements of the theorems. To prove Thm. 1, we assume that q_1, q_2 and $\Pi \in M_{2,n}$,

$\mathcal{U}_\Pi \subset \partial\Omega$ are as in its statement. We will make use of a variant of Alessandrini's identity [A]. For $j = 1, 2$, let $v^{(j)}_{\rho_j}$ be the exact solution to $(\Delta + q_j)v = 0$ constructed above, so that $v^{(j)}_{\rho_j}(x) = e^{\rho_j\cdot x}u^{(j)}(x,\Pi,\rho_j)$, with $u^{(j)} = u^{(j)}_{app} + u^{(j)}_2$. Taking $\rho_1 = \rho, \rho_2 = -\rho$, consider the quantity

$$I = \int_{\partial\Omega} \frac{\partial v^{(1)}_\rho}{\partial n} \cdot v^{(2)}_{-\rho} - v^{(1)}_\rho \cdot \frac{\partial v^{(2)}_{-\rho}}{\partial n} d\sigma.$$

Under the assumption that $v^{(1)}_\rho$ and $v^{(2)}_{-\rho}$ have the same Cauchy data on $\mathcal{U}_\Pi$, I is equal to the integral of the same expression over $\partial\Omega\backslash\mathcal{U}_\Pi$. Observing that

$$\frac{\partial v^{(1)}_\rho}{\partial n} = e^{\rho\cdot x}(\frac{\partial}{\partial n} + (\rho\cdot n(x)))u^{(1)} \text{ and } \frac{\partial v^{(2)}_{-\rho}}{\partial n} = e^{-\rho\cdot x}(\frac{\partial}{\partial n} - (\rho\cdot n(x)))u^{(2)},$$

we see that the exponentials cancel and the integrand of I is

$$= \frac{\partial u^{(1)}}{\partial n} \cdot u^{(2)} - u^{(1)} \cdot \frac{\partial u^{(2)}}{\partial n} + 2(\rho\cdot n(x))u^{(1)}u^{(2)}.$$

Since (1.6) implies that $\text{supp}(u^{(j)}_{app}|\partial\Omega), \text{supp}(\frac{\partial u^{(j)}_{app}}{\partial n}|_{\partial\Omega}) \subset \mathcal{U}_\Pi$ for $|\rho|$ sufficiently large, we have that

$$I = \int_{\partial\Omega\backslash\mathcal{U}_\Pi} \frac{\partial u^{(1)}_2}{\partial n} \cdot u^{(2)}_2 - u^{(1)}_2 \cdot \frac{\partial u^{(2)}_2}{\partial n} + 2(\rho\cdot n(x))u^{(1)}_2 u^{(2)}_2 d\sigma.$$

We estimate

$$
\begin{aligned}
\left|\int_{\partial\Omega\setminus\mathcal{U}_\Pi} \frac{\partial u_2^{(1)}}{\partial n}\cdot u_2^{(2)} d\sigma\right| &\leq \left\|\frac{\partial u_2^{(1)}}{\partial n}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|u_2^{(2)}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\leq \|u_2^{(1)}\|_{H^{\frac{1}{2}}(\partial\Omega)} \cdot \|u_2^{(2)}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\leq C\|u_2^{(1)}\|_{H^1(\Omega)} \cdot \|u_2^{(2)}\|_{H^1(\Omega)} \text{ by Sobolev restriction} \\
&\leq C\|u_2^{(1)}\|_{H^1_t(\mathbb{R}^n)} \cdot \|u_2^{(2)}\|_{H^1_t(\mathbb{R}^n)} \text{ since } \Omega \text{ compact} \\
&\leq C|\rho|^{-2\epsilon} \to 0 \text{ as } |\rho| \to \infty
\end{aligned}
$$

and similarly for the second term. Now note that $|\rho\cdot n(x)| \leq c|\rho|$ since $\partial\Omega$ is Lipschitz, and

$$\|u_2^{(j)}\|_{L^2(\partial\Omega)} \leq \|u_2^{(j)}\|_{H^\sigma(\partial\Omega)} \leq c_\sigma \|u_2^{(j)}\|_{H^{\sigma+\frac{1}{2}}(\Omega)} \leq c'_\sigma |\rho|^{\sigma-\frac{1}{2}-\epsilon}$$

for any $\sigma > 0$, and thus the third term is dominated by $(c'_\sigma)^2|\rho|\cdot|\rho|^{2\sigma-1-2\epsilon} \to 0$ as $|\rho| \to 0$ if we choose $0 < \sigma < \epsilon$.

On the other hand,

$$
\begin{aligned}
I &= \int_{\partial\Omega} \frac{\partial v^{(1)}}{\partial n}\cdot v^{(2)} - v^{(1)} \cdot \frac{\partial v^{(2)}}{\partial n} d\sigma \\
&= \int_\Omega \Delta(v^{(1)})\cdot v^{(2)} - v^{(1)}\cdot\Delta(v^{(2)}) dx \text{ by Green's Thm.} \\
&= \int_\Omega (-q_1 v^{(1)})\cdot v^{(2)} - v^{(1)}\cdot(-q_2 v^{(2)}) dx \\
&= \int_\Omega (q_2 - q_1) v^{(1)} v^{(2)} dx = \int_\Omega (q_2-q_1) u^{(1)} u^{(2)} dx
\end{aligned}
$$

since the exponentials cancel. As $u^{(1)}\cdot u^{(2)} = (u^{(1)}_{app} + u^{(1)}_2)\cdot(u^{(2)}_{app}+u^{(2)}_2)$ and the leading term $u^{(1)}_{app} u^{(2)}_{app} \to d\lambda^\Omega_\Pi$ weakly as $|\rho| \to \infty$ by (1.8), while the remaining terms $\to 0$ since $\|u^{(j)}_{app}\|_{L^2(\Omega)} \leq C$ by (1.5) and $\|u_2^{(j)}\|_{L^2(\Omega)} \leq c|\rho|^{-1-\epsilon}$, we conclude that $I \to R^\Omega_{2,n}\big(q_2 - q_1\big)(\Pi)$ as $|\rho| \to \infty$, finishing the proof of Thm. 1.

Now, to start the proof of Thm. 4 we may use the rotation invariance of Δ and the invariance of $\mathcal{Q}$ under $S^1 = \{e^{i\theta}\}$, and note that it suffices to treat the case[2] $\rho = |\rho|(\vec{e_1} + i\vec{e_2})$, where $\{\vec{e_1}, \dots, \vec{e_n}\}$ is the standard orthonormal

[2]Of course, the length of this element of $\mathcal{Q}$ is $\sqrt{2}|\rho|$, but this is irrelevant for the proofs, and denoting the length of $|\rho|(\vec{e_1}+i\vec{e_2})$ by $|\rho|$ is notationally convenient.

basis for $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$ as $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and similarly $\xi = (\xi', \xi'')$.

If $\Pi \in M_{2,n}$ is parallel to $\text{span}\{\omega_R, \omega_I\} = \text{span}\{\vec{e_1}, \vec{e_2}\} = \mathbb{R}^2 \times \{0\}$, then $\Pi = \text{span}\{\vec{e_1}, \vec{e_2}\} + (0, x_0'')$ for some $x_0'' \in \mathbb{R}^{n-2}$. Given $|\rho| > 1$ and $x_0'' \in \mathbb{R}^{n-2}$, we will define an approximate solution $u(x, \rho, \Pi)$ to $(\Delta_\rho + q(x))u = 0$ on $\mathbb{R}^n$, of the form $u(x, \rho, \Pi) = u_0(x, \rho, \Pi) + u_1(x, \rho, \Pi)$.

For notational convenience, we will usually suppress the dependence on ρ and Π and simply write $u(x) = u_0(x) + u_1(x)$. We will use various cutoff functions χ_j; for j even or odd, χ_j will always denote a function of x' or x'', respectively. Also, $B^m(a; r)$ and $S^{m-1}(a; r)$ will denote the closed ball and sphere of radius r centered at a point $a \in \mathbb{R}^m$.

To define u_0, first fix $\chi_0 \in C_0^\infty(\mathbb{R}^2)$ with $\chi_0 \equiv 1$ on $B^2(0; R)$ for any $R > sup\{|x'| : (x', x'') \in \Omega \text{ for some } x'' \in \mathbb{R}^{n-2}\}$; let $C_0 = \|\chi_0\|_{L^2(\mathbb{R}^2)}$. Secondly, let $\psi_1 \in C_0^\infty(\mathbb{R}^{n-2})$ be radial, nonnegative, supported in the unit ball, and satisfy

$$\int_{\mathbb{R}^{n-2}} (\psi_1(x''))^2 dx'' = 1.$$

Now, for $\beta > 0$ to be fixed later, we let δ be the small parameter $\delta = |\rho|^{-\beta}$ and define

$$\chi_1(x'') = \delta^{-\frac{n-2}{2}} \psi_1 \left(\frac{x' - x_0''}{\delta} \right),$$

so that

$$\|\chi_1\|_{L^2(\mathbb{R}^{n-2})} = \|\psi_1\|_{L^2(\mathbb{R}^{n-2})} = 1, \ \forall\, \delta > 0. \tag{1.9}$$

Set $u_0(x) = u_0(x', x'') = \chi_0(x')\chi_1(x'')$; then u_0 is real, $\|u_0\|_{L^2(\mathbb{R}^n)} = C_0$ and $\|u_0\|_{L^2(\Omega)} \to [\lambda_\Pi(\Pi \cap \Omega)]^{\frac{1}{2}}$ as $\delta \to 0^+$, i.e., as $|\rho| \to \infty$. Note also that $||u_0||_{H^1} \leq c\delta^{-1} = c|\rho|^\beta$, so that $||u_0||_{H^s} \leq c|\rho|^{s\beta}$ for $0 \leq s \leq 1$. Since $\Delta_\rho = \Delta + 2\rho \cdot \nabla = \Delta + 2|\rho|(\vec{e_1} + i\vec{e_2}) \cdot \nabla = \Delta + 4|\rho|\bar{\partial}_{x'}$ and $\rho \perp \mathbb{R}^{n-2}$,

$$\begin{aligned}
(\Delta_\rho + q(x))u_0 &= (\Delta\chi_0) \cdot \chi_1 + 2(\nabla\chi_0) \cdot (\nabla\chi_1) + \chi_0(\Delta\chi_1) \\
&\qquad + 2(\rho \cdot \nabla)(\chi_0)\chi_1 + 2\chi_0(\rho \cdot \nabla)(\chi_1) + q\chi_0\chi_1 \\
&= \chi_0(x')(\Delta_{x''} + q)(\chi_1)(x'') \text{ on } B^2(0; R) \times \mathbb{R}^{n-2},
\end{aligned}$$

the first and fourth terms after the first equality vanishing because $(\rho \cdot \nabla)(\chi_0) = 2\bar{\partial}\chi_0 \equiv 0$ on $B^2(0; R)$, and the second and fifth equalling zero because $\nabla\chi_1 \perp \mathbb{R}^2$.

To define the second term in the approximate solution, $u_1(x)$, we make use of a truncated form of the Faddeev Green function, G_ρ, and an associated projection operator. The operator Δ_ρ has, for $\rho \in \mathcal{Q}$, (full) symbol

$$\sigma(\xi) = -[(|\xi|^2 - 2|\rho|\omega_I \cdot \xi) + i2|\rho|(\omega_R \cdot \xi)], \tag{1.10}$$

and so for $\frac{\rho}{|\rho|} = e_1 + ie_2$, we have

$$\sigma(\xi) = -[(|\xi - |\rho|\vec{e_2}|^2 - |\rho|^2) + i(2|\rho|\xi_1)],$$

which has (full) characteristic variety

$$\begin{aligned} \Sigma_\rho &= \{\xi \in \mathbb{R}^n : \xi_1 = 0, |\xi - |\rho|e_2| = |\rho|\} \\ &= \{0\} \times S^{n-2}((|\rho|, 0, \dots, 0); |\rho|) \subset \mathbb{R}_{\xi_1} \times \mathbb{R}^{n-1}_{\xi_2,\xi''}. \end{aligned} \tag{1.11}$$

The Faddeev Green function is then defined by $G_\rho = (-\sigma(\xi)^{-1})^\vee \in \mathcal{S}'(\mathbb{R}^n)$. We now introduce, for an $\epsilon_0 > 0$ to be fixed later, a tubular neighborhood of Σ_ρ,

$$T_\rho = \{\xi : \mathrm{dist}(\xi, \Sigma_\rho) < |\rho|^{-\frac{1}{2}-\epsilon_0}\}, \tag{1.12}$$

as well as its complement, T_ρ^C, and let χ_{T_ρ}, $\chi_{T_\rho^C}$ be their characteristic functions. Define a projection operator, P_ρ, and a truncated Green function, $\widetilde{G}_\rho$, by

$$\widehat{P_\rho f}(\xi) = \chi_{T_\rho}(\xi) \cdot \widehat{f}(\xi) \text{ and} \tag{1.13}$$

$$(\widetilde{G}_\rho f)^\wedge(\xi) = \chi_{T_\rho^C}(\xi) \cdot [-\sigma(\xi)]^{-1}\widehat{f}(\xi) \tag{1.14}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Note that $\Delta_\rho \widetilde{G}_\rho = I - P_\rho$.

Choose a $\psi_3 \in C_0^\infty(\mathbb{R}^{n-2})$, supported in $B^{n-2}(0;2)$, radial and with $\psi_3 \equiv 1$ on $\mathrm{supp}(\psi_1)$, and set $\chi_3(x'') = \psi_3(\frac{x''-x_0''}{\delta})$. We now define the second term, $u_1(x, \rho, \Pi)$ in the approximate solution by

$$u_1(x) = -\chi_3(x'')\widetilde{G}_\rho((\Delta_\rho + q(x))u_0(x)) \tag{1.15}$$

and set $u(x) = u_0(x) + u_1(x)$. Then u_1 (as well as u_0) is supported in $\{x : \mathrm{dist}(x, \Pi) \leq 2\delta\}$, yielding (1.6). We will see below that $\|u_1\|_{L^2(\Omega)} \leq C|\rho|^{-\epsilon}$ as $|\rho| \to \infty$, so that (1.5) holds as well, so that the first part of (1.9) holds

as well. To start the proof of (1.7), note that

$$
\begin{aligned}
(\Delta_\rho+q)(u_0+u_1) &= (\Delta_\rho+q)u_0-(\Delta_\rho+q)\chi_3\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\
&= (\Delta_\rho+q)u_0-\chi_3(\Delta_\rho+q)\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\
&\quad -[\Delta_\rho+q,\chi_3]\widetilde{G}_\rho((\Delta_\rho+q)u_0)\\
&= (\Delta_\rho+q)u_0-\chi_3(I-P_\rho)(\Delta_\rho+q)u_0-\chi_3 q\widetilde{G}_\rho(\Delta_\rho+q)u_0\\
&\quad -2(\nabla\chi_3\cdot\nabla_{x''})\widetilde{G}_\rho(\Delta_\rho+q)u_0-(\Delta_{x''}\chi_3)\widetilde{G}_\rho(\Delta_\rho+q)u_0\\
&= \chi_3 P_\rho(\Delta_\rho+q)u_0\\
&\quad -[q\chi_3+2(\nabla\chi_3\cdot\nabla_{x''})-(\Delta_{\chi''}\chi_3)]\widetilde{G}_\rho(\Delta_\rho+q)u_0
\end{aligned}
$$

on Ω, since $\chi_3\equiv 1$ on $\mathrm{supp}(\chi_1)$. Now, since $q_1\chi_3\in L^\infty$, $|\nabla\chi_3|\le C\delta^{-1}=c|\rho|^\beta$ and $|\Delta_{x''}\chi_3|\le C\delta^{-2}=c|\rho|^{2\beta}$, (1.7) will follow if we can show that for some $\epsilon>0$,

$$
\begin{aligned}
&(1.16) & \|P_\rho(\Delta_\rho+q)u_0\|_{L^2(\Omega)} &\le C|\rho|^{-\epsilon},\\
&(1.17) & \|\,|D''|\widetilde{G}_\rho(\Delta_\rho+q)u_0\|_{L^2(\Omega)} &\le C|\rho|^{-\beta-\epsilon}, \text{ and}\\
&(1.18) & \|\widetilde{G}_\rho(\Delta_\rho+q)u_0\|_{L^2(\Omega)} &\le C|\rho|^{-2\beta-\epsilon},
\end{aligned}
$$

with C independent of $|\rho|>1$. Before proceeding to prove these, we note that for any $u^{(1)},u^{(2)}$ constructed in this way for the same two-plane Π,

$$
u_0^{(1)}(x)u_0^{(2)}(x)=\chi_0^2(x')\delta^{-(n-2)}\psi_1^2\left(\frac{x''-x_0''}{\delta}\right)\to d\lambda_\Pi^\Omega \text{ in } \Omega
$$

as $\delta\to 0$ by (1.11), while $u_1^{(1)}u_0^{(2)}+u_0^{(1)}u_1^{(2)}+u_1^{(1)}u_1^{(2)}\to 0$ in $L^2(\Omega)$, yielding (1.8). Thus, we are reduced to establishing (1.17–1.19).

2 L^2 estimates

We will first prove (1.17)–(1.19) under the simplifying assumption that $q_1,q_2\in C^{n-1+\sigma}(\overline{\Omega})$ for some $\sigma>0$, turning to the Sobolev space case in Section 3. Start by noting that the desired estimates (1.17)–(1.19) cannot be simply obtained from operator norms; for example, $\|P_\rho\|_{L^2\to L^2}=1$ for all ρ. One needs to make use of the special structure of $(\Delta_\rho+q)u_0$; we first deal with $\Delta_\rho u_0$, leaving $q(x)\cdot u_0$ for the end. So, we will show that $\|P_\rho\Delta_\rho u_0\|_{L^2}\le C|\rho|^{-\epsilon}$, etc. Since $\nabla\chi_0\cdot\nabla\chi_1\equiv 0$,

$$
(2.1)\qquad \Delta_\rho u_0=\chi_0\Delta_{x''}\chi_1+(\Delta_{x'}+4|\rho|\overline{\partial}_{x'})(\chi_0)\cdot\chi_1.
$$

The second term is supported on Ω^c, but P_ρ and $\widetilde{G}_\rho$ are nonlocal operators and we need to control the contribution from this term. However, because $\Delta_{x'}(\chi_0)$ is a fixed, δ-independent element of $C_0^\infty(\mathbb{R}^2)$, this can be handled in the same way as the $q(x)\cdot u_0$ terms of (1.17–1.19), which will be dealt with later. The contribution from $4|\rho|\overline{\partial}\chi_0\cdot\chi_1$ will be handled at the end.

So, for the time being, we are interested in estimating $\|P_\rho(\chi_0(x')\Delta_{x''}\chi_1(x''))\|_{L^2}$, etc. Now, $\Delta_{x''}\chi_1(x'')=\delta^{-2}\chi_5(x'')$, where $\chi_5(x'')=\delta^{-\frac{n-2}{2}}\psi_5\left(\frac{x''-x_0''}{\delta}\right)$ is associated with the radial function $\psi_5=\Delta_{x''}\psi_1$ as χ_1 is associated with ψ_1. Note for future use that $\widehat{\psi}_5$ vanishes to second order at 0. Of course, $\chi_0\in C_0^\infty\Rightarrow\widehat{\chi}_0\in\mathcal{S}(\mathbb{R}^n)$, but looking ahead to estimating the terms involving $q(x)\cdot u_0(x)$, we will now prove the analogues of (1.17–1.19) where P_ρ and $\widetilde{G}_\rho$ act on $\chi_2(x')\Delta\chi_1(x'')$, under the weaker assumption that χ_2 is radial and satisfies the uniform decay estimate

$$(2.2)_\alpha \qquad |\widehat{\chi}_2(\xi)|\le C(1+|\xi|)^{-\alpha}$$

for some $\alpha>0$.

Now, by (1.14) and Plancherel,

$$\begin{aligned}\|P_\rho(\chi_2\Delta\chi_1)\|_{L^2(\Omega)} &\le \|(P_\rho(\chi_2\Delta\chi_1))^\wedge\|_{L^2(\mathbb{R}^n)}\\ &= \|\delta^{-2}|\widehat{\chi}_2(\xi')|\delta^{\frac{n-2}{2}}|\widehat{\psi}_5(\delta\xi'')|\,\|_{L^2(T_\rho)}.\end{aligned}$$

The characteristic variety Σ_ρ, of which T_ρ is a tubular neighborhood, passes through the origin, and we may represent Σ_ρ near O as a graph over the ξ''-plane: $\Sigma_\rho=\Sigma_\rho^s\cup\Sigma_\rho^n\cup\Sigma_\rho^e$, with

$$(2.3)\qquad \begin{aligned}\Sigma_\rho^s &= \left\{\xi_1=0,\xi_2=|\rho|-(|\rho|^2-|\xi''|^2)^{\frac12},|\xi''|\le\frac{|\rho|}{2}\right\}\\ &\simeq \left\{\xi_1=0,\xi_2=\frac{|\xi''|}{2|\rho|},|\xi''|\le\frac{|\rho|}{2}\right\}\end{aligned}$$

a neighborhood of the south pole O,

$$(2.4)\qquad \begin{aligned}\Sigma_\rho^n &= \left\{\xi_1=0,\xi_2=|\rho|+(|\rho|^2-|\xi''|^2)^{\frac12},|\xi''|\le\frac{|\rho|}{2}\right\}\\ &\simeq \left\{\xi_1=0,\xi_2=2|\rho|-\frac{|\xi''|^2}{2|\rho|},|\xi''|\le\frac{|\rho|}{2}\right\}\end{aligned}$$

a neighborhood of the north pole $(0,2|\rho|,0,\dots,0)$, and Σ_ρ^e a neighborhood of the equator $\{\xi\in\Sigma_\rho:\xi_2=|\rho|\}$. We have a corresponding decomposition

$T_\rho = T_\rho^s \cup T_\rho^n \cup T_\rho^e$, where, e.g.,

$$T_\rho^s \simeq \left\{(\xi', \xi'') : \xi' \in B^2\left(\left(0, \frac{|\xi''|^2}{2|\rho|}\right); |\rho|^{-\frac{1}{2}-\epsilon_0}\right), |\xi''| \le \frac{|\rho|}{2}\right\}. \tag{2.5}$$

Recalling that χ_2 and ψ_3 are radial, so are $\widehat{\chi}_2$ and $\widehat{\chi}_3$, and by abuse of notation we consider these as functions of one variable satisfying $(2.2)_\alpha$ and rapidly decreasing, respectively. Thus, using polar coordinates in ξ'',

$$\begin{aligned} \|\widehat{\chi_2\Delta\chi_1}\|^2_{L^2(T_\rho^s)} &\simeq \int_0^{\frac{|\rho|}{2}} \int_{B^2\left(\left(0,\frac{r^2}{2|\rho|}\right);|\rho|^{-\frac{1}{2}-\epsilon_0}\right)} |\widehat{\chi}_2(\xi')|^2 d\xi' \delta^{n-6} |\widehat{\psi}_5(\delta r)|^2 r^{n-3} dr \\ &\simeq \int_0^{\sqrt{2}|\rho|^{\frac{1}{4}}} \int_{B^2((0,0);|\rho|^{-\frac{1}{2}-\epsilon_0})} |\widehat{\chi}_2|^2 d\xi' \delta^{n-6} |\widehat{\psi}_5(\delta r)|^2 r^{n-2} \frac{dr}{r} \\ &\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{4}}}^{\frac{|\rho|}{2}} \left|\widehat{\chi}_2\left(\frac{r^2}{2|\rho|}\right)\right|^2 \cdot |B^2((0,0);|\rho|^{-\frac{1}{2}})| \delta^{n-6} |\widehat{\psi}_5(\delta r)|^2 r^{n-2} \frac{dr}{r}. \end{aligned} \tag{2.6}$$

Since we will be taking $\delta = |\rho|^{-\beta}$ with $\beta < \frac{1}{4}$, if we choose $0 < \epsilon_0 < 2(\frac{1}{4} - \beta)$, then the quantity $|\rho|^{\frac{1}{4}}\delta \to \infty$ as $|\rho| \to \infty$ and so

$$\begin{aligned} \|\widehat{\chi_2\Delta\chi_1}\|^2_{L^2(T_\rho^s)} &\le c\frac{\delta^{-4}}{|\rho|^{1+2\epsilon_0}} \left(\int_0^{\sqrt{2}|\rho|^{\frac{1}{4}}\delta} |\widehat{\psi}_5(r)|^2 r^{n-2} \frac{dr}{r}\right) \\ &\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{4}}\delta}^{\frac{|\rho|}{2}\delta} \left|\widehat{\chi}_2\left(\frac{r^2}{2\delta^2|\rho|}\right)\right|^2 |\widehat{\psi}_5(r)|^2 r^{n-2} \frac{dr}{r} \\ &\le c(\delta^4|\rho|)^{-1}, \end{aligned} \tag{2.7}$$

which is $\le c|\rho|^{-2\epsilon}$ with $\epsilon = \frac{1}{2}(1 - 4\beta) > 0$.

The other contributions to $\|P_\rho \chi_2 \Delta\chi_1\|_{L^2}$, coming from T_ρ^n and T_ρ^e are handled similarly and are even smaller, due to the decrease of $\widehat{\chi}_2$ and $\widehat{\psi}_5$.

We next turn to estimating $\|\,|D''|\widetilde{G}_\rho \Delta_\rho u_0\|_{L^2}$; by the remark above, we may concentrate on the $\chi_2\Delta\chi_1$ term of $\Delta_\rho u_0$. Then

$$\|\,|D''|\widetilde{G}_\rho(\chi_2\Delta\chi_1)\|^2_{L^2(\Omega)} \le \|\,|\xi''|(\sigma(\xi))^{-1}(\chi_2\Delta\chi_1)^\wedge(\xi)\|^2_{L^2(T_\rho^C)}. \tag{2.8}$$

We may cover T_ρ^C by $T_\rho^{C,s} \cup T_\rho^{C,n} \cup T_\rho^{C,e} \cup T_\rho^{C,\infty}$, where

$$T_\rho^{C,s} = \left\{\xi : \xi' \in B^2\left(\left(0, \frac{|\xi''|^2}{2|\rho|}\right); |\rho|^{-\frac{1}{2}-\epsilon_0}\right)^C \cap B^2\left(\left(0, 2|\rho| - \frac{|\xi''|^2}{2|\rho|}\right); \frac{1}{4}|\rho|\right)^C, |\xi''| \le \frac{|\rho|}{2}\right\}, \tag{2.9}$$

$T_\rho^{C,n}$ is defined similarly,

$$T_\rho^{C,e} = \left\{\xi : \frac{|\rho|}{4} < \xi_2 < \frac{7|\rho|}{4}, |\rho|^{-\frac{1}{2}} < \text{dist}(\xi, \Sigma_\rho) < |\rho|, |\xi''| < 2|\rho|\right\} \tag{2.10}$$

and

$$T_\rho^{C,\infty} = \left\{\xi : |\xi| \geq 3|\rho|, |\xi''| \geq \frac{3}{2}|\rho|\right\}. \tag{2.11}$$

One has the lower bounds on σ,

$$|\sigma(\xi)| \geq \begin{cases} C|\rho|\text{dist}(\xi, \Sigma_\rho), & |\xi| \leq 3|\rho| \\ C|\xi|^2, & |\xi| \geq 3|\rho| \end{cases} \tag{2.12}$$

with C (as always) uniform in $|\rho|$. The first inequality in (2.12) follows from noting that $\frac{1}{2}\nabla\sigma(\xi) = (\xi - |\rho|\vec{e_2}) + i(|\rho|\vec{e_1})$, so that $|\nabla\sigma(\xi)| = 2\sqrt{2}|\rho|$ on Σ_ρ, while the second follows from $Re(\sigma(\xi)) = \text{dist}(\xi, |\rho|\vec{e_2})^2 - |\rho|^2$. Using the first estimate in (2.12), we can then dominate the contribution to the right side of (2.8) from the region $T_\rho^{C,s}$ by

(2.13)
$$\delta^{n-6} \int_{|\xi''| \leq \frac{|\rho|}{2}} \int_{B^2\left(\left(0, \frac{|\xi''|^2}{2|\rho|}\right); |\rho|^{-\frac{1}{2}-\epsilon_0}\right)^C} |\rho|^{-2} \left|\xi' - \frac{|\xi''|^2}{2|\rho|}\vec{e_2}\right|^{-2} |\widehat{\chi}_2(\xi')|^2 d\xi' |\xi''|^2 |\widehat{\psi}_5(\delta\xi'')|^2 d\xi''.$$

The inner integral is the convolution

$$|\rho|^{-2} \left(|\widehat{\chi}_2|^2 *_{\mathbb{R}^2} \frac{\chi\{|\xi'| \geq |\rho|^{-\frac{1}{2}-\epsilon_0}\}}{|\xi'|^2} \right)\Bigg|_{\xi' = \frac{|\xi''|^2}{2|\rho|}\vec{e_2}}.$$

An elementary calculation shows that, for $\widehat{\chi}_2$ satisfying $(2.2)_\alpha$ for some $0 < \alpha < 1$, and any $0 < a < 1$,

$$|\widehat{\chi}_2|^2 *_{\mathbb{R}^2} \frac{\chi\{|\xi'| \geq a\}}{|\xi'|^2} \leq \begin{cases} C_1(1 + \log(a^{-1})), & |\xi'| \leq 1 \\ C_2|\xi'|^{-2} + C_3|\xi'|^{-2\alpha} \log\left(\frac{|\xi'|}{a}\right), & |\xi'| \geq 1, \end{cases} \tag{2.14}$$

so that, taking $a = |\rho|^{-\frac{1}{2}-\epsilon_0}$ and $|\xi'| = \frac{|\xi''|^2}{2|\rho|}$, the inner integral in (2.13) is

$$\leq \begin{cases} C_1|\rho|^{-2}\log|\rho|, & 0 < |\xi''| \leq \sqrt{2}|\rho|^{\frac{1}{2}} \\ C_2|\xi''|^{-4} + C_3|\rho|^{2\alpha-2}|\xi''|^{-4\alpha} \log\left(\frac{|\xi''|^2}{2|\rho|^{\frac{1}{2}-\epsilon_0}}\right), & \sqrt{2}|\rho|^{\frac{1}{2}} \leq |\xi''| \leq \frac{|\rho|}{2}. \end{cases}$$

Employing polar coordinates in ξ'' and rescaling by δ, we see that (2.13) is

$$\begin{aligned}
\leq\ & C_1\delta^{-6}|\rho|^{-2}\log|\rho|\int_0^{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}|\widehat{\psi}_5(r)|^2 r^n\frac{dr}{r}\\
&+C_2\delta^{-2}\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta}|\widehat{\psi}_5(r)|^2 r^{n-4}\frac{dr}{r}\\
&+C_3\delta^{4\alpha-4}|\rho|^{2\alpha-2}\log|\rho|\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta}|\widehat{\psi}_5(r)|^2 r^{n-2-4\alpha}\frac{dr}{r}.
\end{aligned}$$

With $\delta=|\rho|^{-\beta}$, $\beta<\frac{1}{4}$, $|\rho|^{\frac{1}{2}}\delta\to\infty$ as $|\rho|\to\infty$, and thus we estimate this for any $N>0$ (using the rapid decay of $\widehat{\psi}_5$) by

$$C_1|\rho|^{6\beta-2}\log|\rho|+C_2\delta^{-2}(|\rho|^{\frac{1}{2}}\delta)^{-N}+C_3|\rho|^{(4-4\alpha)\beta+2\alpha-2}\log|\rho|(|\rho|^{\frac{1}{2}}\delta)^{-N},$$

the first term of which will be less than the desired $|\rho|^{-2\beta-2\epsilon}$, for any $\alpha>0$, if $\beta<\frac{1}{4}$ and $\epsilon=\frac{1}{2}(1-4\beta)$; the second and third terms are rapidly decaying simply because $\beta<\frac{1}{2}$.

Moving ahead for the moment to (1.18), the contribution to $\|\widetilde{G}_\rho\chi_2\Delta\chi_1\|_{L^2}^2$ (which we want $\leq C|\rho|^{-4\beta-2\epsilon}$) from $T_\rho^{C,s}$ is handled in the same fashion, the only differences being the absence of the multiplier $|D''|^\wedge=|\xi''|$ on the left and the improved gain we are demanding on the right. Taking these into account, we need to control

$$\begin{aligned}
(2.15)\quad & C_1\delta^{-4}|\rho|^{-2}\log|\rho|\int_0^{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}|\widehat{\psi}_5(r)|^2 r^{n-2}\frac{dr}{r}\\
&+C_2\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{1}{2}|\rho|\delta}|\widehat{\psi}_5(r)|^2 r^{n-6}\frac{dr}{r}\\
&+C_3\delta^{4\alpha-2}|\rho|^{2\alpha-2}\log|\rho|\int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{1}{2}|\rho|\delta}|\widehat{\psi}_5(r)|^2 r^{n-4-4\alpha}\frac{dr}{r}\\
&\leq C_1\delta^{-4}|\rho|^{-2}\log|\rho|+C_2(|\rho|^{\frac{1}{2}}\delta)^{-N}+C_N\delta^{4\alpha-2}|\rho|^{2\alpha-2}\log|\rho|(|\rho|^{\frac{1}{2}}\delta)^{-N},
\end{aligned}$$

and this is $\leq C|\rho|^{-4\beta-2\epsilon}$ provided $\beta<\frac{1}{4}$, $\epsilon<\frac{1}{2}(1-4\beta)$ and N is sufficiently large.

The contributions to (1.18) from $T_\rho^{C,n}$ and $T_\rho^{C,e}$ are handled similarly. To treat the contribution from $T_\rho^{C,\infty}$, we use the second estimate in (2.12) and

calculate (for (1.18)

$$
\begin{aligned}
(2.16)\quad & \| \, |\xi''| (\sigma(\xi))^{-1} (\chi_2 \Delta \chi_1)^\wedge(\xi) \|^2_{L^2(T_\rho^{C,\infty})} \\
&\le C \iint_{|\xi| \ge 3|\rho|} \delta^{n-6} |\widehat{\chi}_2(\xi')|^2 |\widehat{\psi}_5(\delta\xi'')|^2 \frac{|\xi''|^2 d\xi' d\xi''}{|\xi|^4} \\
&\le C \left(\int_{|\xi''| \le |\rho|} \delta^{n-6} |\rho|^{-2\alpha-2} |\widehat{\psi}_5(\delta\xi'')|^2 |\xi''|^2 d\xi'' \right. \\
&\quad \left. + \int_{|\xi''| \ge |\rho|} \delta^{n-6} |\widehat{\psi}_5(\delta\chi'')|^2 |\xi''|^{-2\alpha} d\xi'' \right) \\
&= C \left(\delta^{-6} |\rho|^{-2\alpha-2} \int_0^{|\rho|\delta} |\widehat{\psi}_5(r)|^2 r^n \frac{dr}{r} \right. \\
&\quad \left. + \delta^{2\alpha-4} \int_{|\rho|\delta}^\infty |\widehat{\psi}_5(r)|^2 r^{n-2-2\alpha} \frac{dr}{r} \right) \\
&\le C(\delta^{-6}|\rho|^{-2\alpha-2} + \delta^{2\alpha-4}(|\rho|\delta)^{-N}), \qquad \forall\, N > 0,
\end{aligned}
$$

which, for $\delta = |\rho|^{-\beta}$ and N large is $\le C|\rho|^{-2\beta-2\epsilon}$ provided $\beta < \frac{1}{4}$ and $\epsilon < \alpha + 1 - 4\beta$. A similar analysis holds for the $T_\rho^{C,\infty}$ contribution to (1.19).

We now turn to controlling the $q(x)u_0(x)$ terms in (1.17)–(1.19), as well as the contributions from the $\Delta(\chi_0) \cdot \chi_1$ term in (2.1). Note that since $q(x)$ is $C^{n-1+\sigma}$ (for some $\sigma > 0$), $q(x)$ has an extension (see, e.g., [St70,Ch.6]) to a $C^{n-1+\sigma}$ function of compact support on $\mathbb{R}^n$, which we also denote by q. The restriction of q to any $\Pi \in M_{2,n}$ is still $C^{n-1+\sigma}$.

Let $\{D_t : 0 < t < \infty\}$ be the one-parameter group of partial dilations on $\mathcal{S}'(\mathbb{R}^{n^*})$,

$$(D_t f)(\xi', \xi'') = t^{n-2} f(\xi', t\xi''),$$

which, for $f, g \in L^1$, satisfy $\int_{\mathbb{R}^n} D_t f d\xi = \int_{\mathbb{R}^n} f d\xi$ and $D_t(f * g) = D_t f * D_t g$. Then

$$
\begin{aligned}
(2.17)\widehat{qu_0}(\xi) &= \widehat{q} * \widehat{u}_0(\xi) = D_\delta(D_{\delta^{-1}}\widehat{q}) * \delta^{-\frac{n-2}{2}} D_\delta(\widehat{\chi}_0(\xi')\widehat{\psi}_1(\xi'')e^{ix_0''\cdot\xi''}) \\
&= D_\delta(D_{\delta^{-1}}(\widehat{q}) * \delta^{-\frac{n-2}{2}} \widehat{\chi}_0 \widehat{\psi}_1 e^{ix_0''\cdot\xi''}).
\end{aligned}
$$

Now, as $\delta = |\rho|^{-\beta} \to 0$, $D_{\delta^{-1}}(\widehat{q}) = \delta^{-(n-2)}\widehat{q}(\xi', \delta\xi'')$ converges weakly to the singular measure

$$Q(\xi') \otimes \delta(\xi'') = Q(\xi')d\xi', \tag{2.18}$$

where $Q(\xi') = \int_{\mathbb{R}^{n-2}} \widehat{q}(\xi', \xi'') d\xi''$; note that $q \in C^{n-1+\gamma}$ implies that the integral defining Q converges and Q satisfies $(2.2)_{1+\gamma}$. Letting $F(\xi) = \widehat{\chi}_0(\xi')\widehat{\psi}_1(\xi'')e^{ix_0''\cdot\xi''}$, it follows from (2.17) that

$$
\begin{aligned}
(2.19)\widehat{qu_0}(\xi) &= D_\delta(D_{\delta^{-1}}(\widehat{q}) * \delta^{-\frac{n-2}{2}} F) \\
&= D_\delta((Qd\xi') * \delta^{-\frac{n-2}{2}} F) + D_\delta((D_{\delta^{-1}}\widehat{q} - Qd\xi') * \delta^{-\frac{n-2}{2}} F).
\end{aligned}
$$

If we define $\widehat{\chi}_4(\xi') = Q *_{\mathbb{R}^2} \widehat{\chi}_0(\xi')$, then $\widehat{\chi}_4$ also satisfies condition $(2.2)_{1+\gamma}$ (and thus $(2.2)_{\alpha'}$ for $0 < \alpha' < 1$, so that (2.14) can be applied), and the first term in (2.19) is

$$
(2.20) \qquad D_\delta((Qd\xi') * \delta^{-\frac{n-2}{2}} F) = \widehat{\chi}_4(\xi')\delta^{\frac{n-2}{2}}\widehat{\psi}_1(\delta\xi'')e^{i\delta x_0''\cdot\xi''}.
$$

Thus, the contributions to $\|P_\delta(qu_0)\|_{L^2}$, $\|\,|D''|\widetilde{G}_\rho(qu_0)\|_{L^2}$ and $\|\widetilde{G}_\rho(qu_0)\|_{L^2}$ from the first term in (2.19) may be handled as the main $\chi_2\Delta\chi_1$ term was earlier, with the obvious absence of the factor δ^{-2}. To control the contributions from the second term in (2.19), we use the elementary

Lemma 5 *Let $\varphi(x)$, $f(x)$ be functions on $\mathbb{R}^m$ such that $\varphi(x)$, $|x|\varphi(x)$, $f(x)$ and $|\nabla f(x)|$ are in $L^1(\mathbb{R}^m)$. Then, $\forall\ \epsilon > 0$*

$$
\begin{aligned}
\left|\left(\epsilon^{-m}\varphi\left(\frac{x}{\epsilon}\right)\right.\right. &- \left.\left.\left(\int_{\mathbb{R}^m}\varphi dy\right)\delta(x)\right) * f(x)\right| \\
&\leq C_m(\|\varphi\|_{L^1} + \|\,|x|\varphi\|_{L^1}) \cdot (\|f\|_{L^\infty(B(0;|x|-1))} + \|\nabla f\|_{L^\infty(B(x;1))}) \cdot \epsilon.
\end{aligned}
$$

Applying this for $\epsilon = \delta$, $\xi' \in \mathbb{R}^2$ fixed, and using $F \in \mathcal{S}$, $|\widehat{q}(\xi)| \leq C(1+|\xi|)^{-(n-1+\gamma)}$, we find that, $\forall\ N > 0$

$$
(2.21) \qquad |(D_{\delta-1}(\widehat{q}) - Qd\xi') * F(\xi)| \leq C_N(1+|\xi'|)^{-\gamma}(1+|\xi''|)^{-N}\delta.
$$

Hence, the second term in (2.19) is $\leq C_N\delta^{\frac{n}{2}}(1+|\xi'|)^{-\gamma}(1+|\delta\xi''|)^{-N}$ and this allows the contributions to (1.17)–(1.19) to be dealt with as the $\chi_2\Delta_{x''}\chi_1$ term was before.

Finally, we need to establish the estimates (1.17–1.19) for the $4|\rho|\overline{\partial}\chi_0$ term in (2.1); thus, we need to show

$$
\begin{aligned}
(2.22) \qquad \|P_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2} &\leq C|\rho|^{-1-\epsilon}, \\
(2.23) \qquad \|\,|D''|\widetilde{G}_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2} &\leq C|\rho|^{-1-\beta-\epsilon}, \text{ and} \\
(2.24) \qquad \|\widetilde{G}_\rho(\overline{\partial}\chi_0 \cdot \chi_1)\|_{L^2} &\leq C|\rho|^{-1-2\beta-\epsilon},
\end{aligned}
$$

for some $\epsilon > 0$. Using the fact that $\widehat{\overline{\partial}\chi_0}(\xi')$ is rapidly decreasing and vanishes to first order at $\xi' = 0$, we may replace (2.6) with

$$
\begin{aligned}
\|\widehat{\overline{\partial}\chi_0\chi_1}\|^2_{L^2(T^s_\rho)} \quad &\simeq \quad \int_0^{\frac{|\rho|}{2}} \int_{B^2\left(\left(0,\frac{r^2}{2|\rho|}\right);|\rho|^{-\frac{1}{2}-\epsilon_0}\right)} |\widehat{\overline{\partial}\chi_0}(\xi')|^2 d\xi' \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-3} dr \\
&\leq c_N \Big(\int_0^{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}} |\rho|^{-2-4\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r} \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}}^{\sqrt{2}|\rho|^{\frac{1}{2}}} (\frac{r^2}{2|\rho|})^2 |\rho|^{-1-2\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r} \\
&\quad + \int_{\sqrt{2}|\rho|^{\frac{1}{2}}}^{\frac{|\rho|}{2}} (\frac{r^2}{2|\rho|})^{-N} |\rho|^{-1-2\epsilon_0} \delta^{n-2} |\widehat{\psi}_1(\delta r)|^2 r^{n-2} \frac{dr}{r} \Big) \\
&\leq c_N \Big(|\rho|^{-2-4\epsilon_0} \int_0^{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}\delta} |\widehat{\psi}_1|^2 r^{n-2} \frac{dr}{r} \\
&\quad + |\rho|^{-3-2\epsilon_0} \delta^{-4} \int_{\sqrt{2}|\rho|^{\frac{1-2\epsilon_0}{4}}\delta}^{\sqrt{2}|\rho|^{\frac{1}{2}}} |\widehat{\psi}_1|^2 r^{n+2} \frac{dr}{r} \\
&\quad + |\rho|^{-1-2\epsilon_0+N} \delta^{2N} \int_{\sqrt{2}|\rho|^{\frac{1}{2}}\delta}^{\frac{|\rho|}{2}\delta} |\widehat{\psi}_1|^2 r^{n-2-2N} \frac{dr}{r} \Big) \\
&\leq c_N \quad \big(|\rho|^{-2-4\epsilon_0} + |\rho|^{-3-2\epsilon_0+4\beta} (|\rho|^{\frac{1-2\epsilon_0}{4}-\beta})^{-N'} \\
&\quad + |\rho|^{-1-2\epsilon_0+N-2N\beta-N'(\frac{1}{2}-\beta)} \big)
\end{aligned} \tag{2.25}
$$

for any $N, N' \geq 0$. As before, the contributions from T^n_ρ and T^e_ρ are handled similarly. Since $\epsilon_0 < \frac{1}{2} - 2\beta$, if N' is chosen large enough this yields (2.23) with $\epsilon \leq 2\epsilon_0$, which is weaker than the previously imposed $\epsilon < \frac{1}{2}(1-4\beta)$.

The desired estimates (2.23),(2.24) are even easier and hold for any $\beta < \frac{1}{2}$. The contribution to (2.24) from $T^{C,s}_\rho$ is controlled as in (2.13), but with the factor δ^{n-2} and with the $\widehat{\chi_2}$ in the integrand replaced by $\widehat{\overline{\partial}\chi_0}$; this is then dominated in the same manner as below (2.14). The $T^{C,s}_\rho$ contribution to (2.25) is estimated as in (2.15), but with the absence of the δ^{-4}. All other contributions are dealt with similarly.

This concludes the proof of Thm.4 for the case of potentials in the Hölder class $C^{n-1+\sigma}(\overline{\Omega}), \sigma > 0$. The restrictions on β and ϵ that we ahve needed are that $\beta < \frac{1}{4}$ and $\epsilon < \frac{1}{2}(1-4\beta)$.

3 Remarks

(i) The proof of Thm. 4 needs to be slightly modified if we assume that the potential $q(x)$ belongs to the Sobolev space $H^{\frac{n}{2}+\sigma}(\Omega)$ for some $\sigma > 0$. Since $\partial\Omega$ is Lipschitz, such a $q(x)$ can, by the Calderón extension theorem, be extended to be in $H^{\frac{n}{2}+\sigma}(\mathbb{R}^n)$. Again denoting the extension by q, one has by Cauchy-Schwarz

$$(3.1)\qquad \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^{n-2}}(1+|\xi''|)|\hat{q}(\xi',\xi'')|d\xi''\Big)^2(1+|\xi'|)^\sigma d\xi' \leq c\big(\|q\|_{\frac{n}{2}+\sigma}\big)^2$$

Thus, Q as in (2.18) belongs to $L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')$, so that $\widehat{\chi}_4 = Q *_{\mathbb{R}^2} \widehat{\chi}_0 \in L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')\cap L^\infty$. Replacing the uniform decay estimate $(2.2)_\alpha$ with

$$(3.2)_\sigma \qquad \widehat{\chi}_2 \in L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')$$

will allow us to handle the first term in (2.19). Furthermore, if for ξ' fixed, we let $\phi(\cdot)=\widehat{q}(\xi',\cdot)$ in Lemma 5, then $\phi(\xi'')$ and $|\xi''|\phi(\xi'')$ are in $L^1(\mathbb{R}^{n-2})$ with norms (as functions of ξ') in $L^2(\mathbb{R}^2;(1+|\xi'|)^\sigma d\xi')$, and so the second term in (2.19) is $\leq c_N\widehat{\chi_6}(\xi')(1+|\delta\xi''|)^{-N}, \forall N$, with $\widehat{\chi_6}$ satisfying condition $(3.2)_\sigma$. So, we are reduced to repeating the analysis of Section 2 with $(2.2)_\alpha$ replaced by $(3.2)_\sigma$. The decay of $\widehat{\chi_2}$ was used in only two places in the argument. In (2.14), under $(3.2)_\sigma$, we have the same estimate except for the absence of $|\xi'|^{-2\alpha}$; however, this loss is absorbed into terms rapidly decreasing in $|\rho|^{\frac{1}{2}}\delta = |\rho|^{\frac{1}{2}-\beta}$ where (2.14) is used. On the other hand, in (2.16) we may estimate the inner integral by

$$(3.3)\qquad \begin{aligned}\int_{|\xi'|\geq 2|\rho|}|\widehat{\chi_2}(\xi')|^2\frac{d\xi'}{(|\xi'|^2+|\xi''|^2)^2} &\leq \textstyle\int_{\mathbb{R}^2}|\widehat{\chi_2}|^2\frac{d\xi'}{(1+|\xi'|)^\sigma|\xi'|^4}\\ &\leq c|\rho|^{-4-\sigma} \text{ if } |\xi''|\leq\rho\end{aligned}$$

and

$$(3.4)\qquad \int_{\mathbb{R}^2}|\widehat{\chi_2}(\xi')|^2\frac{d\xi'}{(|\xi'|^2+|\xi''|^2)^2} \leq c|\xi''|^{-4} \text{ if } |\xi'|\geq\rho,$$

so that

$$
(3.5) \qquad \| \, |\xi''|(\sigma(\xi))^{-1}(\chi_2\Delta\chi_1)^\wedge(\xi)\|^2_{L^2(T^{C,\infty}_\rho)}
$$

$$
\begin{aligned}
&\leq C \left(\int_{|\xi''|\leq|\rho|} \delta^{n-6}|\rho|^{-4-\sigma}|\widehat{\psi}_5(\delta\xi'')|^2|\xi''|^2 d\xi'' + \int_{|\xi''|\geq|\rho|} \delta^{n-6}|\widehat{\psi}_5(\delta\chi'')|^2|\xi''|^{-2} d\xi'' \right) \\
&= C \left(\delta^{-6}|\rho|^{-4-\sigma} \int_0^{|\rho|\delta} |\widehat{\psi}_5(r)|^2 r^n \frac{dr}{r} \right. \\
&\quad \left. + \delta^{-2} \int_{|\rho|\delta}^\infty |\widehat{\psi}_5(r)|^2 r^{n-4} \frac{dr}{r} \right) \\
&\leq C_N(\delta^{-6}|\rho|^{-4-\sigma} + \delta^{-2}(|\rho|\delta)^{-N}) \\
&= C_N \left(|\rho|^{6\beta-4-\sigma} + |\rho|^{2\beta}(|\rho|^{\beta-\frac{1}{2}})^N \right), \qquad \forall N,
\end{aligned}
$$

which is $\leq c|\rho|^{-2\beta-\epsilon}$ for N sufficiently large, since $\beta < \frac{1}{2}$. The restrictions on β and ϵ are as before.

(ii) The construction of the approximate solutions given by Thm. 4 may be generalized by taking χ_0 to be an arbitrary analytic function of $z = x_1+ix_2$, defined on a domain $\Pi\cap\Omega \subset\subset \Omega' \subset \Pi$. Since $\overline{\partial}\chi_0 = \Delta_{x'}\chi_0 \equiv 0$ on Ω, the resulting $u = u_0+u_1$ is still an approximate solution in the sense of Thm. 4, except that (1.8) no longer applies. Thus, Thm. 1 can be strengthened to conclude that $(q_1-q_2)|_\Pi$ is orthogonal in $L^2(\Pi\cap\Omega, d\lambda_\Pi)$ to the Bergman space $A^2(\Pi\cap\Omega)$ of square-integrable holomorphic functions on $\Pi\cap\Omega$. Furthermore, by repeating the construction using $\overline{\rho} = \frac{1}{\sqrt{2}}|\rho|(\omega_R - i\omega_I)$, which induces the conjugate complex structure on Π, for which the $\overline{\partial}$ operator equals the ∂ operator induced by ρ, we obtain that $(q_1 - q_2)|_\Pi$ is also orthogonal to the conjugate Bergman space $\overline{A}^2(\Pi \cap \Omega)$ of anti-holomorphic functions. (The analogue of this in two dimensions was obtained in [SU87b].) It would be interesting to make further use of this information.

(iii) To obtain variants of Thm. 1 establishing smaller sets of uniqueness in $\partial\Omega$, it might be useful to use approximate solutions associated to different two-planes. For this, it seems necessary to construct approximate solutions with much thinner supports, i.e., to overcome the restriction $\beta < \frac{1}{4}$ in Thm. 4. Such an improvement might also be useful in extending the results to $q_j \in L^\infty$.

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